

AD-A106 789

AIR FORCE INST OF TECH WRIGHT-PATTERSON AFB OH F/G 17/2.1
ANALYSIS OF MAXIMIN MATCHED-FILTER DESIGN FOR COMMUNICATION THR--ETC(U)
1981 C J ALSTON
AFIT-CI-81-13T

UNCLASSIFIED

NL

1 * 1
COUNT

1

END
DATE
FILMED
12-81
DTIC

AD A106789

DTIC FILE COPY

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER 81-13T	2. GOVT ACCESSION NO. AD-A106789	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) Analysis of Maximin Matched-Filter Design for Communication Through a Nonlinearly Distorting Channel.		5. TYPE OF REPORT & PERIOD COVERED THESIS/DISSERTATION
7. AUTHOR(s) Capt Clifton J. Alston		6. PERFORMING ORG. REPORT NUMBER
9. PERFORMING ORGANIZATION NAME AND ADDRESS AFIT STUDENT AT: University of Illinois		8. CONTRACT OR GRANT NUMBER(s)
11. CONTROLLING OFFICE NAME AND ADDRESS AFIT/NR WPAFB OH 45433		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBER
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office) <i>AFIT/NR</i>		12. REPORT DATE 1981
		13. NUMBER OF PAGES 34
		15. SECURITY CLASS. (of this report) UNCLASS
16. DISTRIBUTION STATEMENT (of this Report) APPROVED FOR PUBLIC RELEASE; DISTRIBUTION UNLIMITED (14) AFIT CI-81-13T		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)		
18. SUPPLEMENTARY NOTES APPROVED FOR PUBLIC RELEASE: IAW AFR 190-17		20 OCT 1981 Fredric C. Lynch FREDRIC C. LYNCH, Major, USAF Director of Public Affairs Air Force Institute of Technology (ATC) Wright-Patterson AFB, OH 45433
19. KEY WORDS (Continue on reverse side if necessary and identify by block number)		
20. ABSTRACT (Continue on reverse side if necessary and identify by block number) ATTACHED		

DD FORM 1 JAN 73 1473

EDITION OF 1 NOV 65 IS OBSOLETE

LB

UNCLASS

012200

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

81 10 26 149

ABSTRACT

↓

This thesis describes a study of the problem of antipodal signalling through a nonlinearly distorting channel. Maximin design criteria are presented, and general expressions are derived for the robust filter and its performance in terms of the eigenfunctions and eigenvalues of certain additive noise autocorrelations. In particular, the results of baseband detection in triangular-kernel noise, ideally bandlimited noise, and the Wiever noise processes are presented.

↑

~

AFIT RESEARCH ASSESSMENT

81-13T

The purpose of this questionnaire is to ascertain the value and/or contribution of research accomplished by students or faculty of the Air Force Institute of Technology (AFIT). It would be greatly appreciated if you would complete the following questionnaire and return it to:

AFIT/NR

Wright-Patterson AFB OH 45433

RESEARCH TITLE: Analysis of Maximin Matched-Filter Design for Communication Through a Nonlinearly Distorting Channel

AUTHOR: Capt Clifton J. Alston

RESEARCH ASSESSMENT QUESTIONS:

1. Did this research contribute to a current Air Force project?
() a. YES () b. NO
2. Do you believe this research topic is significant enough that it would have been researched (or contracted) by your organization or another agency if AFIT had not?
() a. YES () b. NO
3. The benefits of AFIT research can often be expressed by the equivalent value that your agency achieved/received by virtue of AFIT performing the research. Can you estimate what this research would have cost if it had been accomplished under contract or if it had been done in-house in terms of manpower and/or dollars?
() a. MAN-YEARS _____ () b. \$ _____
4. Often it is not possible to attach equivalent dollar values to research, although the results of the research may, in fact, be important. Whether or not you were able to establish an equivalent value for this research (3. above), what is your estimate of its significance?
() a. HIGHLY SIGNIFICANT () b. SIGNIFICANT () c. SLIGHTLY SIGNIFICANT () d. OF NO SIGNIFICANCE
5. AFIT welcomes any further comments you may have on the above questions, or any additional details concerning the current application, future potential, or other value of this research. Please use the bottom part of this questionnaire for your statement(s).

NAME

GRADE

POSITION

ORGANIZATION

LOCATION

STATEMENT(s):

By _____	
Distribution/Availability _____	
Dist	Availability _____
A	Special _____

81 10 26 149

FOLD DOWN ON OUTSIDE - SEAL WITH TAPE

AFIT/NR
WRIGHT-PATTERSON AFB OH 45433

OFFICIAL BUSINESS
PENALTY FOR PRIVATE USE. \$300



NO POSTAGE
NECESSARY
IF MAILED
IN THE
UNITED STATES

BUSINESS REPLY MAIL

FIRST CLASS PERMIT NO. 73236 WASHINGTON D.C.

POSTAGE WILL BE PAID BY ADDRESSEE

AFIT/ DAA
Wright-Patterson AFB OH 45433



FOLD IN

ANALYSIS OF MAXIMIN MATCHED FILTER DESIGN
FOR COMMUNICATION THROUGH A
NONLINEARLY DISTORTING CHANNEL

BY

CLIFTON J. ALSTON

B.S., University of Illinois, 1976

THESIS

Submitted in partial fulfillment of the requirements
for the degree of Master of Science in Electrical Engineering
in the Graduate College of the
University of Illinois at Urbana-Champaign, 1981

Urbana, Illinois

ANALYSIS OF MAXIMIN MATCHED FILTER DESIGN FOR COMMUNICATION
THROUGH A NONLINEARLY DISTORTING CHANNEL

Clifton J. Alston
Department of Electrical Engineering
University of Illinois at Urbana-Champaign, 1981

ABSTRACT

This thesis describes a study of the problem of antipodal signalling through a nonlinearly distorting channel. Maximin design criteria are presented, and general expressions are derived for the optimum filter and its performance in terms of the eigenfunctions and eigenvalues of certain additive noise autocorrelation functions. In particular, results for baseband detection in triangular-kernel noise, ideally bandlimited noise, and Wiener noise processes are presented.

ACKNOWLEDGEMENT

The author would like to express his sincere appreciation to the following people whose contributions helped make this work possible. Thanks goes to Dr. H. V. Poor, the advisor on this project, who offered invaluable theoretical guidance and encouragement. The author is also indebted to Sue Killian for drafting the figures contained in this thesis, and to Phyllis Young who offered skill and patience in typing the preliminary, the text, and the references pages of this thesis.

TABLE OF CONTENTS

CHAPTER	Page
I. GENERAL DEVELOPMENTS.....	1
II. THE TRIANGULAR-KERNEL NOISE PROCESS.....	7
III. THE IDEALLY BANDLIMITED NOISE PROCESS.....	19
IV. THE WIENER NOISE PROCESS.....	26
V. SUMMARY AND SUGGESTIONS.....	33
REFERENCES.....	34

CHAPTER I

GENERAL DEVELOPMENTS

This thesis is a study of the demodulation of antipodal signals transmitted through a nonlinearly distorting channel. The motivation for this work is the related research effort by Poor, reported in [1] and [2]. General expressions will be drawn freely from this literature.

An antipodal signalling problem involves the discrimination between two mutually exclusive events. To illustrate, consider the following pair of hypotheses for an observation $Z \equiv \{Z(t); 0 \leq t \leq T\}$:

$$H_0: Z(t) = \eta(t) - s(t) ; \quad 0 \leq t \leq T$$

versus

$$H_1: Z(t) = \eta(t) + s(t) ; \quad 0 \leq t \leq T, \quad (1.1)$$

where $\{s(t); 0 \leq t \leq T\}$ is the known deterministic signal, and $\{\eta(t); 0 \leq t \leq T\}$ is a sample function of a zero-mean additive Gaussian noise process.

To discriminate between H_0 and H_1 in (1.1), consider the decision rule:

$$\gamma(h; z) = \begin{cases} 1 & ; \text{ if } \langle h, z \rangle \geq 0 \\ -1 & ; \text{ if } \langle h, z \rangle < 0 , \end{cases} \quad (1.2)$$

where $h \equiv \{h(t); 0 \leq t \leq T\}$ denotes the impulse response of a linear time-invariant filter, and the operation $\langle h, z \rangle$ is given by

$$\langle h, z \rangle = \int_0^T h(T-t)z(t)dt . \quad (1.3)$$

For a value of $\gamma(h;z) = 1$, the decision rule indicates the choice H_1 , while $\gamma(h;z) = -1$ will indicate the choice H_0 . Assuming H_0 and H_1 are equally likely to occur, the probability of error using decision rule $\gamma(h;z)$ is given by the equation

$$p_e(h) = 1 - \Phi[\langle h, s \rangle / [\langle h, R_N h \rangle]^{1/2}], \quad (1.4)$$

where $\Phi[x]$ is the unit normal cumulative distribution function

$$\Phi[x] = (2\pi)^{-1/2} \int_{-\infty}^x \exp[-u^2/2] du, \quad (1.5)$$

$\langle h, s \rangle$ is as in (1.3), and

$$\langle h, R_N h \rangle = \int_0^T h(T-t) \int_0^T R_N(t, u) h(T-u) du dt, \quad (1.6)$$

where $R_N \equiv \{R_N(t, u); 0 \leq t; u \leq T\}$ is the known autocorrelation function of the additive noise process $\{\eta(t); 0 \leq t \leq T\}$.

The probability of error $p_e(h)$ is a primary measure of system performance for digital data communications. For optimal design with respect to this measure, signal detection is assumed to be accomplished by a linear receiver. Synchronous, or coherent, detection in an additive Gaussian noise background requires a matched-filter, or equivalent correlation detection, to achieve minimum probability of error for fixed signal and noise conditions. The matched-filter impulse response $h(t)$ is known to be the solution to the integral equation

$$s(t) = \int_0^T R_N(t, \tau) h(T-\tau) d\tau, \quad (1.7)$$

a Fredholm equation of the first kind. Here the functions $s(t)$ and $R_N(t, \tau)$ are as previously described.

The choice of antipodal signalling also minimizes probability of error, since this signal design becomes optimal for general binary receivers by maximizing signal-to-noise ratio (SNR). The signal interval and the receiver observation interval are also assumed to coincide.

The nonlinear channel distortion is modelled to affect the signal structure alone, not the noise covariance. Examples of this phenomenon include transmitter or receiver generated noise, improper phase locking, channel fading, and multiple transmission paths. It is usually convenient to model all distortions in the communications channel, and to assume an ideal transmitter and receiver. This channel distortion motivates the following model for the received signal class \mathcal{A} as found in the work by Poor [1]:

$$\mathcal{A} = \{s \in L_2[0, T] : \int_0^T |s(t) - s^0(t)|^2 dt \leq \Delta\}, \quad (1.8)$$

where $\{s^0(t); 0 \leq t \leq T\}$ is a known nominal signal model, $L_2[0, T]$ denotes the square-integrable functions on the interval $[0, T]$, and Δ is the degree of channel distortion.

To design a suitable filter for detection within the distortion model of (1.8), we consider a maximin design criteria as in [1], i.e., we seek a solution to

$$\max_h \min_{s \in \mathcal{A}} \{ \langle h, s \rangle / [\langle h, R_N h \rangle]^{1/2} \}. \quad (1.9)$$

A solution to (1.9) will have the largest possible minimum output signal-to-noise ratio and thus its worst-case performance will be the best possible within the tolerances set by the uncertainties in the signal structure. Such a filter is known as a robust, or optimally stable, matched filter.

Given the problem of interest, with a known noise autocovariance function $\{R_N\}$ and a signal structure as given in (1.8), the robust filter solution is that impulse response $h^R(t)$ which is the solution to the equation

$$s^0 = (R_N + \sigma_0 \underline{I})h^R. \quad (1.10)$$

This is a general result as given in [2], where s^0 is the nominal signal waveform, \underline{I} denotes the identity operator on $L_2[0, T]$, and σ_0 is a non-negative constant determined by the equation

$$\sigma_0^2 \int_0^T |h^R(t)|^2 dt = \Delta. \quad (1.11)$$

Further, it is shown that the impulse response $h^R(t)$ is the matched-filter corresponding (or matched) to the least favorable signal $s^2(t) \in \mathcal{J}$ given by

$$s^L(t) = s^0(t) - \sigma_0 h^R(t), \quad (1.12)$$

and it follows in [2] that the worst filter performance over the class of signals \mathcal{J} and noise $\{R_N\}$ is

$$\begin{aligned} \max_{s(t) \in \mathcal{J}} p_e(h^R(t)) &= 1 - \Phi\left\{\left[\min_{s(t) \in \mathcal{J}} \langle h^R, s \rangle / [\langle h^R, R_N h^R \rangle]^{1/2}\right]\right\} \\ &= 1 - \Phi\left\{[\langle s^L, h^R \rangle]^{1/2}\right\}. \end{aligned} \quad (1.13)$$

Note that (1.10) specifies the robust filter, and (1.13) specifies the worst performance of this filter. Poor [1] points out that since the identity operator \underline{I} corresponds to white noise, the effect of distortion of the type of (1.10) is equivalent to the effect of adding white noise of spectral height σ_0 to the communications channel.

Equation (1.10) is written explicitly as

$$s^0(t) = \sigma_0 h^R(T-t) + \int_0^T R_N(t,u) h^R(T-u) du; \quad (0 \leq t \leq T), \quad (1.14)$$

which is a Fredholm equation of the second kind. Properties of this equation as well as solution techniques are found in the literature [3]. This thesis will employ the method of Hilbert and Schmidt. In particular, if $R_N(t,u)$ is continuous on $[0,T]^2$, then a continuous, square-integrable solution for $h^R(t)$ will always exist.

Moreover, $R_N(t,u)$ has a Mercer expansion on $[0,T]^2$ given by

$$R_N(t,u) = \sum_{N=1}^{\infty} \lambda_N \psi_N(t) \psi_N(u), \quad (1.15)$$

with uniform convergence on $[0,T]^2$. The constants $\{\lambda_N; N = 1, 2, 3, \dots\}$ and the functions $\{\psi_N(t); n = 1, 2, 3, \dots\}$ are the eigenvalues and eigenfunctions, respectively, of $\{R_N\}$; they are the solutions to the homogeneous equation

$$\lambda \psi(t) = \int_0^T R_N(t,u) \psi(u) du; \quad 0 \leq t \leq T, \quad (1.16)$$

with the $\psi_N(t)$'s being orthonormal; i.e.,

$$\int_0^T \psi_N(u) \psi_M(u) du = \begin{cases} 1, & N = M \\ 0, & N \neq M. \end{cases} \quad (1.17)$$

It is desirable to obtain series solutions to the quantities of interest which will represent a unique solution to (1.14). In Lovitt [3], it is shown that the unique solution $h^R(t)$ is given by the series

$$h^R(T-t) = \sigma_0 [s^0(t) - \sum_{n=1}^{\infty} c_N (1 + \sigma_0 / \lambda_N)^{-1} \psi_N(t)]; \quad 0 \leq t \leq T, \quad (1.18)$$

where c_N is the component of $s^0(t)$ along $\psi_N(t)$,

$$c_N = \int_0^T s^0(t) \psi_N(t) dt ; \quad N = 1, 2, 3, \dots \quad (1.19)$$

Combining equations (1.18) and (1.11), it can be shown that the equation specifying σ_0 is

$$\int_0^T |s^0(t)|^2 dt - \sum_{n=1}^{\infty} c_N^2 (1 + 2\sigma_0/\lambda_N) (1 + \sigma_0/\lambda_N)^{-2} = \Delta . \quad (1.20)$$

The least-favorable signal of (1.12) becomes

$$s^L(t) = \sum_{n=1}^{\infty} c_N (1 + \sigma_0/\lambda_N)^{-1} \psi_N(t); \quad 0 \leq t \leq T , \quad (1.21)$$

and the quantity specifying the worst filter performance in (1.13) is given by

$$\langle h^R, s^L \rangle = \sigma_0^{-1} \left[\int_0^T |s^0(t)|^2 dt - \sum_{n=1}^{\infty} c_N^2 (1 + \sigma_0/\lambda_N)^{-1} - \Delta \right] . \quad (1.22)$$

These quantities of interest can be determined when the eigenfunctions and eigenvalues of the noise autocorrelation are known; subsequently, the robust filter and its performance will be specified by equations (1.18) through (1.22). The following chapters will investigate robust filter performance when the noise process $\{R_N\}$ is determined to be: 1) triangular kernel noise, 2) ideally bandlimited noise, and 3) the Wiener noise process.

CHAPTER II

THE TRIANGULAR KERNEL NOISE PROCESS

The "triangular" kernel noise process has been shown by Papoulis [4] to be a model for random binary transmission. The autocorrelation function is given by

$$N_0^{-1} R_N(t, u) = \begin{cases} 0 & \text{if } |t-u| > T_N \\ 1 - |t-u|/T_N & \text{if } |t-u| < T_N \end{cases} \quad (2.1)$$

As an example of channel contamination by extraneous signals, such noise pulses may have pulsewidths less than, equal to, or greater than the binary signal of interest; i.e., T_N in (2.1) is the pulsewidth of an unwanted random binary signal in a communications channel and $T_s \equiv \{T_s \neq T_N\}$ represents the coincident interval of the robust filter and the binary signal of interest.

Papoulis [4] and Thomas [6] have calculated directly from the binary pulse structure the autocorrelation function of triangular noise processes. Their procedures are readily applied to the case $T_s > T_N$ for calculation of $\{R_N\}$; however, for a Mercer expansion of $R_N(t, u)$ on $[0, T_N]$, the filter solution of (1.18) cannot be expressed in terms of eigenfunctions and eigenvalues of $\{R_N\}$ with absolute and uniform convergence assured on the larger interval $[0, T_s]$. Hence, the case $T_s > T_N$ is not considered in this study.

Kailath [5] examined in depth the triangular kernel for $T_s < T_N$, as an example of kernels with nonrational spectra. In particular, the power spectral density function for (2.1) is given by

$$S(f) = N_0 T_N \operatorname{sinc}^2(f T_N), \quad (2.2)$$

where N_0 is the noise power and $f_c = T_N^{-1}$ is the bandwidth. To derive the eigenvalues and eigenfunctions of $\{R_N\}$, define the sequence $\{\beta_n; n = 1, 2, 3, \dots\}$ by

$$(N-1)\pi < \beta_N T_s < N\pi, \quad N = 1, 2, 3, \dots, \quad (2.3)$$

and

$$\tan \frac{1}{2} \beta_N T_s = \pm [T_s / (2 - T_s)] [\frac{1}{2} \beta_N T_s]^{-1} \quad (2.4)$$

where $T_N \triangleq 1$, and thus $0 < T_s < 1$. Then the eigenvalues of (2.1) are given by

$$\lambda_N = 2 N_0 / \beta_N^2, \quad N = 1, 2, 3, \dots; \quad (2.5)$$

and the corresponding orthonormal eigenfunctions are

$$\varphi_N(t) = \varphi_n(t) / \left[\int_0^T |\varphi_n(t)|^2 dt \right]^{1/2}, \quad 0 \leq t \leq T_s, \\ N = 1, 2, 3, \dots \quad (2.6)$$

where

$$\varphi_N(t) = K_1 \sin(\beta_N t) + \cos(\beta_N t); \quad 0 \leq t \leq T_s, \\ N = 1, 2, 3, \dots \quad (2.7)$$

and

$$K_1 = [T_s / (2 - T_s)] [\frac{1}{2} \beta_N T_s]^{-1}.$$

Figure 2.1 justifies the approximation implicit in the above derivation

$$\beta_N T_s \approx \{N/2\} 2\pi, \quad N = 1, 2, 3, \dots \quad (2.8)$$

where $\{x\}$ denotes the greatest integer less than or equal to x .

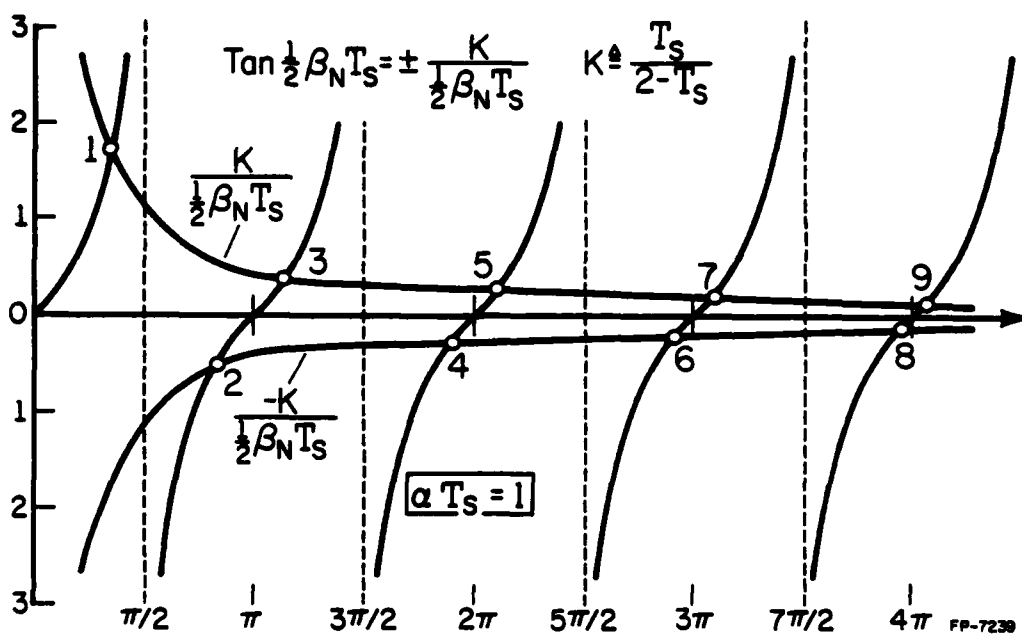


Figure 2.1 Graphical Solution of Transcendental Equation

Figure 2.1. Graphical solution of transcendental equation.

Straightforward computation using (2.4) yields

$$\int_0^T |\varphi_N(t)|^2 dt \cong T_s/2 [1 + \cos(\beta_N T_s) \sin(2\beta_N T_s)/(2\beta_N T_s)] [1 + K_1^2];$$

$$N = 1, 2, 3, \dots \quad (2.9)$$

It then follows to solve equations (1.18) through (1.22) for a nominal signal $s^0(t)$ and a given degree of distortion Δ to determine the associated robust filter and its performance. Several root-finding techniques are available to solve equation (2.4); however, in the case where $T_s/T_N \ll 1$ (narrow-bandwidth noise), the approximation in (2.8) derived from the graphical solution in Fig. 2.1 becomes highly accurate, and the two sequences $\{\beta_N\}$ and $\{\lambda_N\}$ have the following approximations:

$$\beta_N \sim \begin{cases} 2[T_s(2 - T_s)]^{-\frac{1}{2}} & N = 1 \\ \{N/2\} 2\pi/T_s - (-1)^N [2/(2 - T_s)] [\{N/2\}\pi]^{-1} & ; N > 1 \end{cases} \quad (2.10)$$

$$\lambda_N \sim \begin{cases} \frac{1}{2} N_0 T_s (2 - T_s) & ; N = 1 \\ \frac{1}{2} N_0 [T_s / (\{N/2\}\pi)]^2 & ; N > 1 \end{cases} \quad (2.11)$$

As an illustration of robust filter development, consider the nominal baseband signal

$$s^0(t) = E_0 \quad ; \quad 0 \leq t \leq T$$

$$E_0 > 0 \quad (2.12)$$

From (1.19) we have

$$\begin{aligned}
c_N &= \int_0^T E_0 \varphi_N(t) dt \\
&\cong E_0 (1 + K_1^2) \sin(\beta_N T) \beta_N^{-1} / \left[\int_0^T |\varphi_N(t)|^2 dt \right]^{1/2} \quad N = 1, 2, 3, \dots \quad (2.13)
\end{aligned}$$

Note that (2.7) can be rearranged via (2.4) to yield

$$\varphi_N(t) = \cos \beta_N (t - T_s/2) / \cos(\beta_N T_s/2) \quad (2.14)$$

The robust filter is then found from (1.18), specifically

$$\begin{aligned}
h^R(T-t) &= h^R(t) \\
&= E_0 \sigma_0^{-1} [1 - (1 + K_1^2) \sum_{n=1}^{\infty} \gamma_N \cos \beta_N (t - T_s/2)]; \quad 0 \leq t \leq T_s \quad (2.15)
\end{aligned}$$

where

$$\gamma_N = \sin(\beta_N T_s) (\beta_N (1 + \sigma_0/\lambda_N))^{-1} / \int_0^T |\varphi_N(t)|^2 dt \quad N = 1, 2, 3, \dots \quad (2.16)$$

The robust filter performance for triangular-kernel noise is plotted in Figures 2.4 and 2.5. These measures can be compared with the filter performance for the Gauss-Markov noise condition as the distortion in the signal structure is increased. Results from the Gauss-Markov study by Poor [1] are plotted in Figures 2.2 and 2.3. An examination of the quantities in equations (2.15) and (2.16) reveals that the filter impulse response, normalized to $\sigma_0 E_0^{-1} h^R(t)$, depends chiefly on parameters $(\sigma_0/N_0 T_s)$ and bandwidth-pulsewidth product $(T_s T_N^{-1} \equiv \alpha T_s$; recall T_N is defined as unity in this development). The distortion factor $(\sigma_0/N_0 T_s)$ is plotted versus signal distortion in Figure 2.4, for the cases $\alpha T_s = 0.1$ and 1.0.

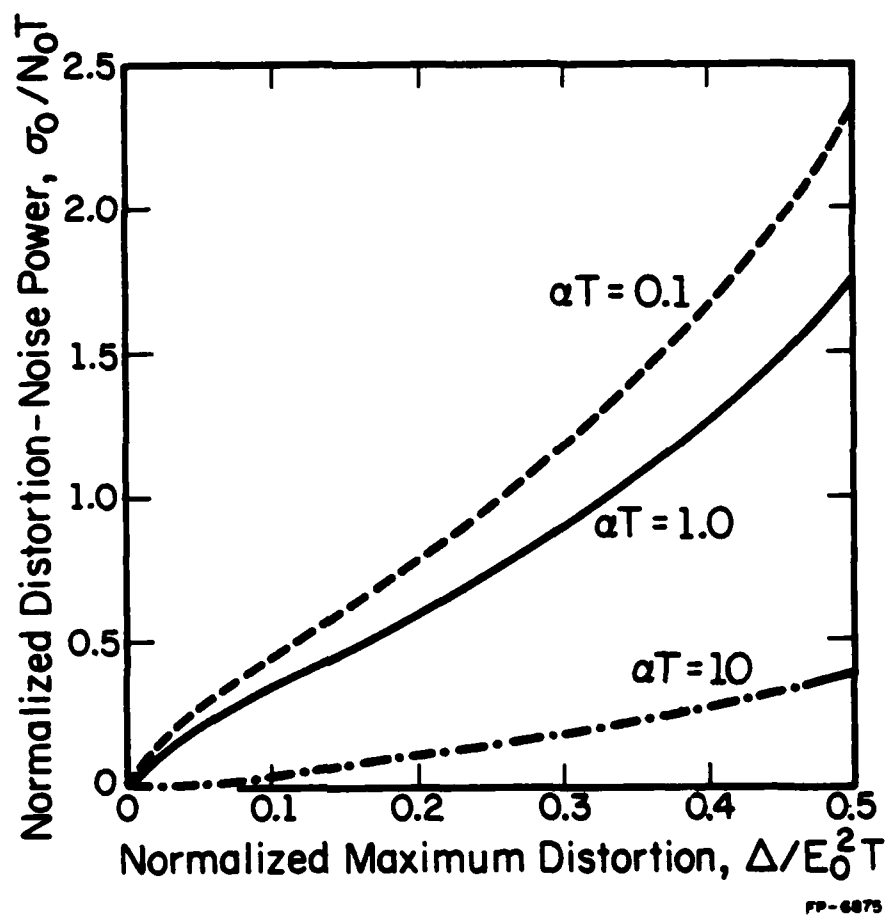


Figure 2.2. Distortion factor versus maximum signal distortion for the Gauss-Markov noise model.

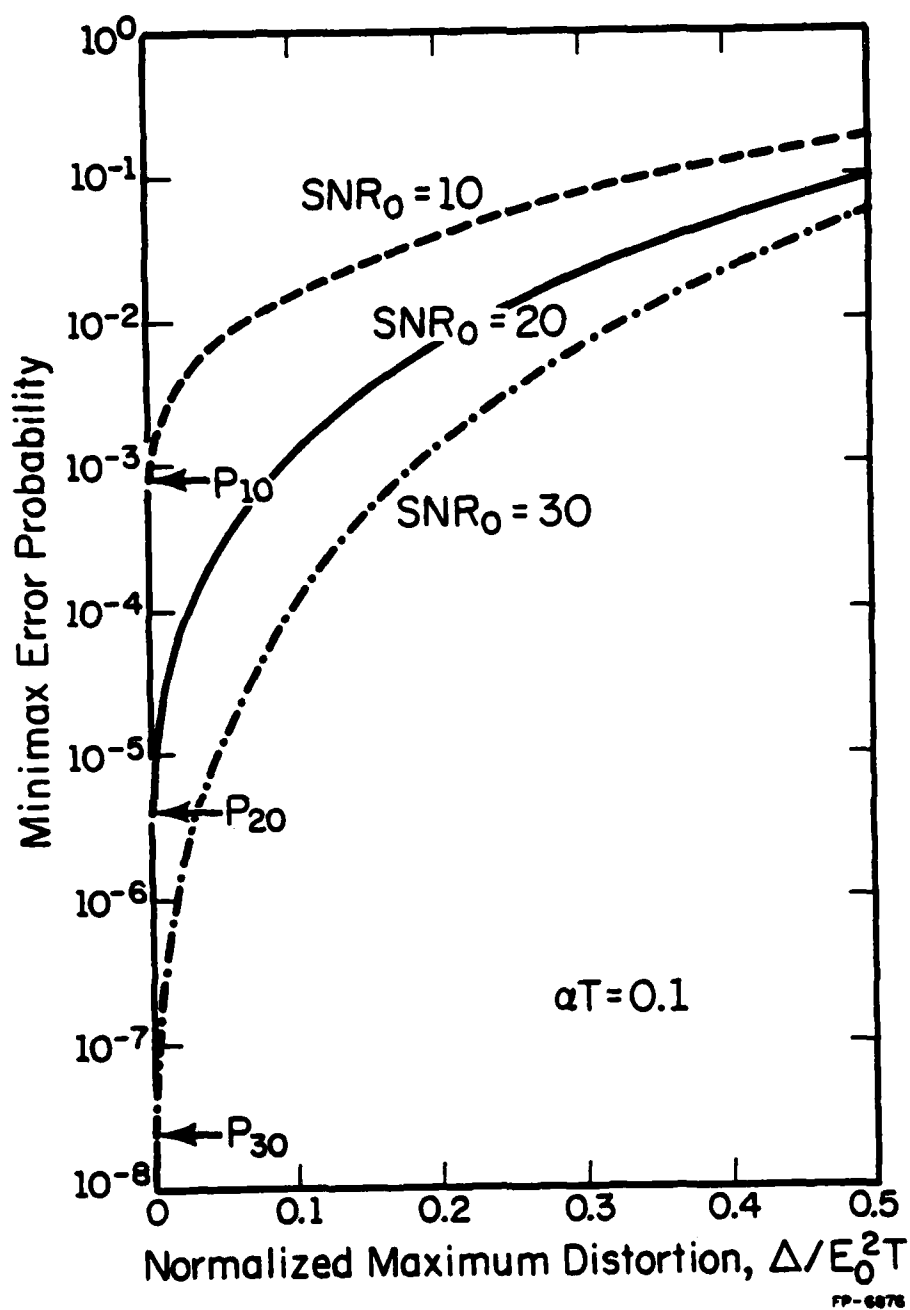


Figure 2.3. Maximin error probability versus maximum distortion for narrowband Gauss-Markov noise.

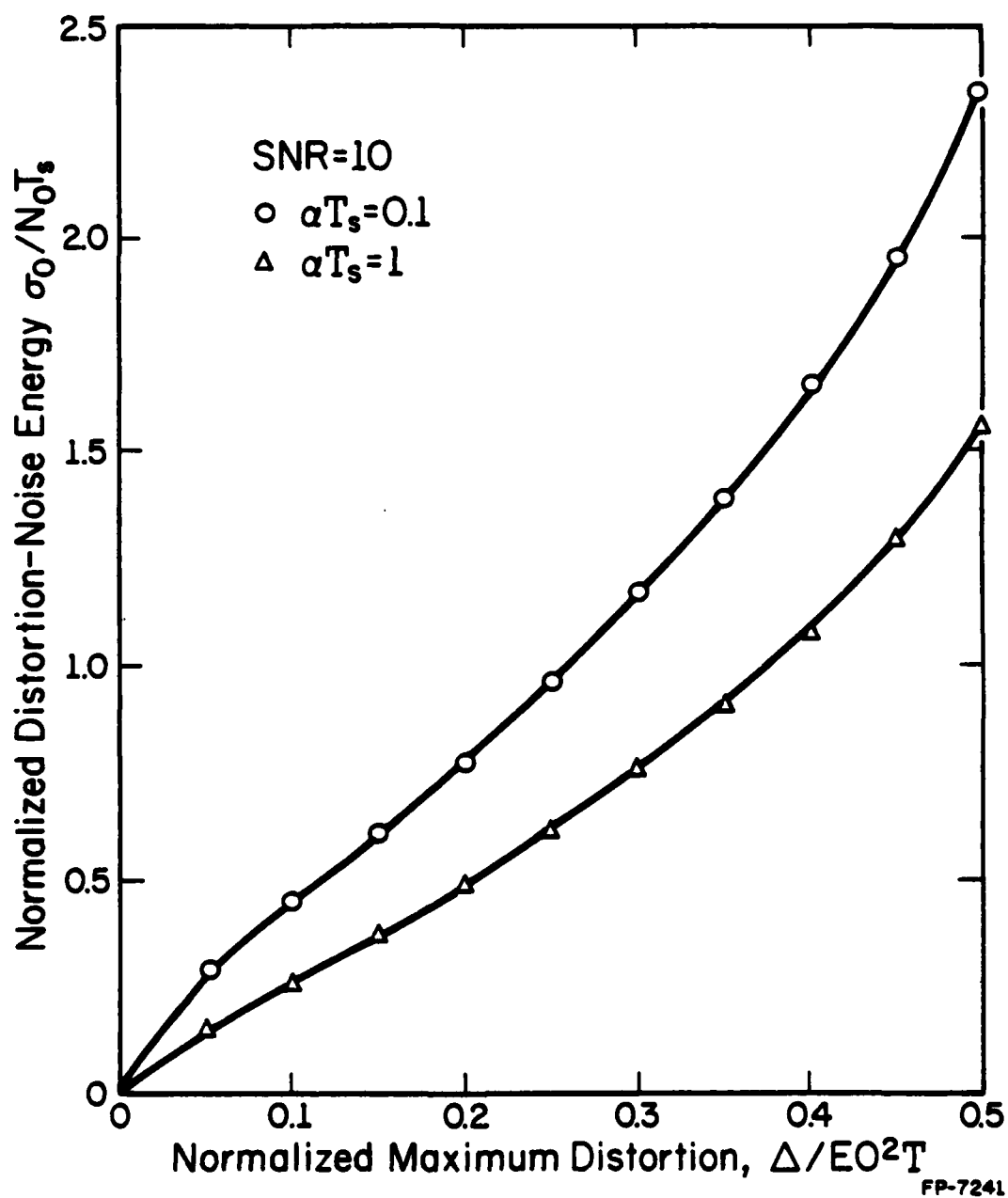


Figure 2.4. Distortion factor versus maximum signal distortion for triangular kernel noise.

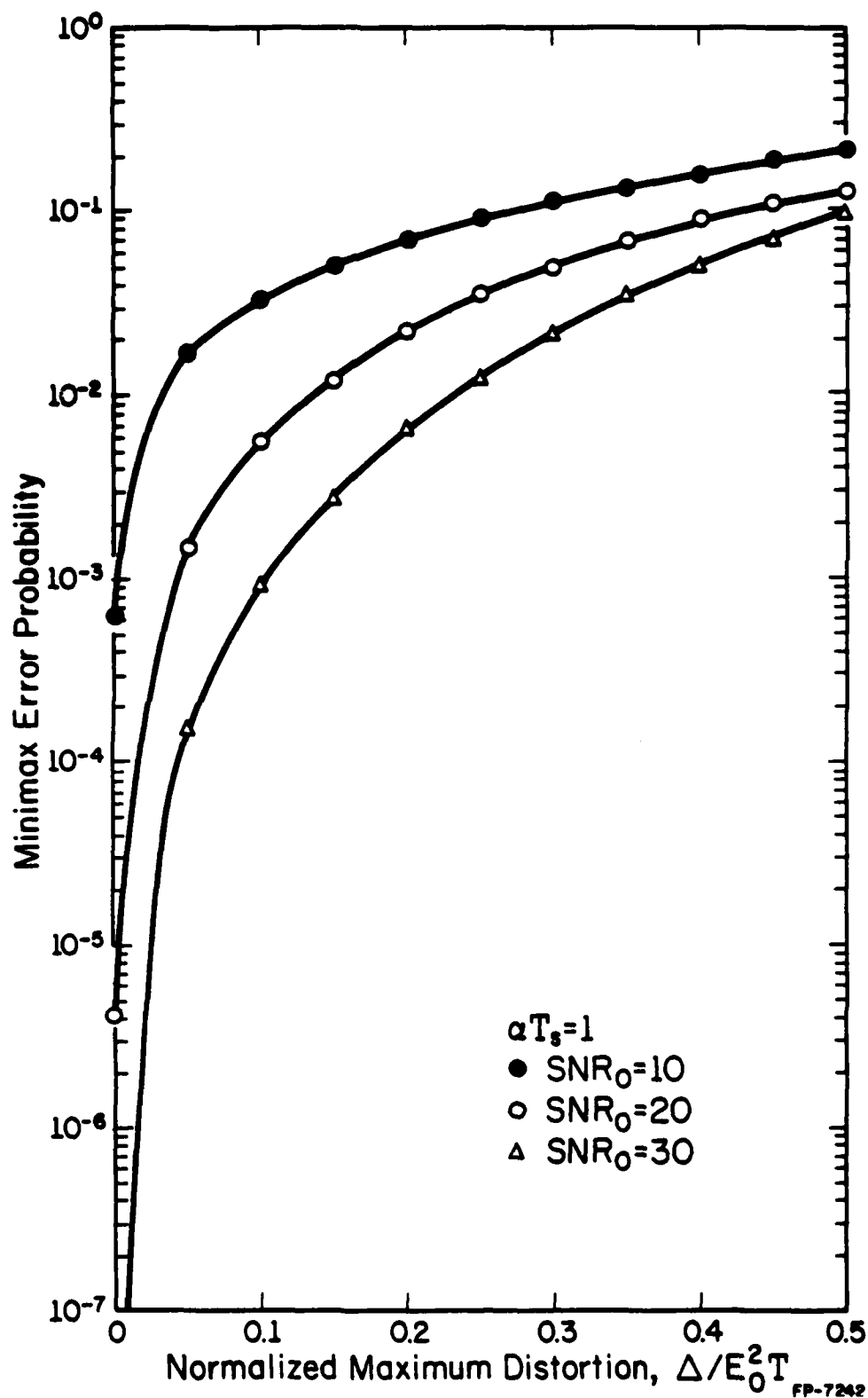


Figure 2.5. Maximin error probability versus maximum signal distortion for the triangular-kernel noise.

The robust filter is calculated via (2.15) for all conditions of (αT_s) and $(\Delta/E_0^2 T_s)$. In all cases, $h^R(t)$ is symmetric about $t = T_s/2$, as predicted by (2.14). The impulse response is lower at this midpoint than at the endpoints, but this amplitude differential in $h^R(t)$ is reduced as $(\Delta/E_0^2 T_s)$ increases. This reduction of endpoint singularities is consistent with solution techniques for integral equations, where the introduction of singularities into the kernel $R_N(t,u)$ (physically, a white noise component in the noise) reduce or eliminate the need for singularities in a continuous square-integrable solution such as $h^R(t)$.

To illustrate this concept, Kailath [5] has shown that the solution to (1.7), with $\{R_N(t,u)\}$ being the triangular kernel, the signal of interest being defined as in (2.12), and with $\Delta = 0$, is an impulse response consisting entirely of singularities at the endpoints.

$$h^R(t) = A \delta(t) + A \delta(t - T_s); \quad 0 \leq t \leq T_s \quad (2.17)$$

where A is a positive constant determined by boundary conditions. Thus, the receiver disregards the signal over the entire observation interval except at the endpoints of the interval. This surprising result is more clearly understood when the triangular kernel is viewed as the model for random binary communication. The filter solution (2.17) is then seen to minimize the variance of the noise process; i.e., this impulse response achieves optimal signal-to-noise ratio, assuming zero white noise component ($\Delta = 0$).

At the opposite extreme, as the channel distortion increases without bound, the slightly lower response of $h^R(t)$ at $t = T_g/2$ approaches the maximum amplitude at the endpoints. The former singularities are decreasing in amplitude as $h^R(t)$ becomes essentially a constant on the interval $[0, T_g]$; i.e., $h^R(t)$ approaches the matched filter solution for detection in a predominantly white noise environment

$$h^R(t) \rightarrow k s^0(T-t) = k E_0; \quad 0 \leq t \leq T_g, \quad (2.18)$$

where k is a scale factor irrelevant to threshold detection. Thus, for both extremes of signal distortion, the robust filter computed by (2.15) is seen to be the optimal impulse response for maximum signal-to-noise ratio (SNR).

The maximin error probability (P_e) is a filter performance measure equivalent to worst case SNR. The robust filter $\max_{s(t) \in \mathcal{S}} P_e(h^R)$, as given by (1.13) and (1.22), is plotted in Figure 2.5 for the case $\alpha T_g = 0.1$ and values of $SNR_0 = 10, 20$, and 30 . SNR_0 is defined as

$$SNR_0 = \lim_{\Delta \rightarrow 0} [\langle h^R, s^L \rangle] = \sum_{n=1}^{\infty} c_n^2 / \lambda_n \quad (2.19)$$

and

$$P_e(SNR_0) = 1 - \phi[(SNR_0)^{\frac{1}{2}}] \quad (2.20)$$

The robust filter performance for the triangular-kernel noise, measured by P_e , is qualitatively similar to the results in the Gauss-Markov noise cases. Performance is seriously degraded for small values of channel distortion; this supports the suggestion by Poor [1] that this detection procedure is very sensitive to L_2 distortion.

Computations to produce Figures 2.4 and 2.5 were performed by digital computer, which required the truncation of infinite series $\{\lambda_N\}$ and $\{\psi_N(t)\}$. To minimize truncation error, the quantity $[\int_0^T |s^0(t)|^2 dt]$ is approximated by $\sum_{n=1}^N (c_n)^2$, and (1.22) becomes

$$\langle h^R, s^L \rangle' = \sigma^{-1} \left[\sum_{n=1}^N c_n^2 (1 + \lambda_n / \sigma_0)^{-1} - \Delta \right]. \quad (2.21)$$

For sufficiently large values of N , all significant eigenvalues are included in the computations, and the error is inconsequential. The estimate in (2.21) is less than the true value of $\langle h^R, s^L \rangle$ for all values of N , thus providing an upper bound for probability of error. The difference term for the estimate is the summation

$$\langle h^R, s^L \rangle - \langle h^R, s^L \rangle' = \sum_{n=N+1}^{\infty} c_n^2 (1 + \lambda_n / \sigma_0)^{-1}. \quad (2.22)$$

This term is made sufficiently small by choosing the proper value of N .

The singular nature of the zero distortion filter, as seen in (2.17), causes the detection to be sensitive to small amounts of distortion. Helstrom [7] notes that this singularity can occur when the noise autocorrelation function has no singular (or white noise) component. Such sensitivity should be more pronounced as the noise processes under consideration become more unlike the purely white noise case. A study which assumes $\{R_N(t, u)\}$ to be ideally band limited white noise provides a striking example of a robust filter's increasing sensitivity.

CHAPTER III

THE IDEALLY BANDLIMITED NOISE PROCESS

A process is considered ideally bandlimited (or ideal low-pass) if it has a power spectral density function defined such that

$$S_X(w) = \begin{cases} S_0 & , \quad |w| < \alpha \\ 0 & , \quad |w| > \alpha \end{cases} \quad (3.1)$$

The corresponding covariance function of this process is given by

$$R_N(t,u) = S_0 \frac{\sin \alpha(t-u)}{\pi(t-u)} \quad ; \quad (3.2)$$

t, u are defined on the interval $[-T/2, T/2]$; the time interval has been shifted to simplify notation. In the literature [8], Slepian and Pollak describe a countably infinite sequence of bandlimited functions $\{\psi_N(t); N = 1, 2, 3, \dots\}$ with the properties:

1) In the interval $t \in [-T/2, T/2]$, the terms of the sequence are orthogonal and complete in the class of complex valued functions which are defined and square integrable in the interval $-T/2 \leq t \leq T/2$, such that

$$-T/2 \int_{-T/2}^{T/2} \psi_M(t) \psi_N(t) dt = \begin{cases} \lambda_N & , \quad M = N \\ 0 & , \quad M \neq N; M, N = 1, 2, 3, \dots \end{cases} \quad (3.3)$$

2) For all values of t , real or complex,

$$\lambda_N \psi_N(t) = \int_{-T/2}^{T/2} \frac{\sin \alpha(t-u)}{\pi(t-u)} \psi_N(u) du; N = 1, 2, 3, \dots \quad (3.4)$$

If the noise structure under consideration is to be second-order stationary, with uniform spectral density in $(-\alpha, \alpha)$ and zero elsewhere, then equation (3.4) becomes the Karhunen-Loeve representation of that bandlimited white noise. This representation has an expansion in terms of the eigenfunctions $\{\psi_N(t); n = 1, 2, 3, \dots\}$ and eigenvalues $\{\lambda_i; i = 1, 2, 3, \dots\}$ as defined in (3.3) and (3.4) on the interval $-T/2 \leq t \leq T/2$.

The equation (3.4) has been investigated, and its solutions, called prolate spheroidal wave functions, have been tabulated. These eigenfunctions are seen to depend on the product $c = \alpha T/2$, or $2c = \alpha T$. Note that, since the solutions are defined on a fixed time interval $[-T/2, T/2]$, a change in the constant c denotes a change in the bandwidth of the noise process. It will be seen that a progressive study of the aforementioned singular nature of a zero distortion filter is possible as the robust filter performance is determined for $c = 0.5, 1.0, 2.0$, and 4.0 .

Numerical computations for this study, as specified by (1.18) through (1.22), will be in terms of the first four eigenvalues and eigenfunctions only, but the properties of the above series of bandlimited functions reduce the effects of truncation errors. In particular, we have that the model for the signal class defined in (1.8), and the assumed noise process are real signals, and as such are both timelimited and bandlimited. This consideration is justified in work [11] by Slepian. It follows from Property 1 that representation of real signals by the bandlimited functions $\{\psi_N(t)\}$ will achieve goodness of fit in the interval $[-T/2, T/2]$; i.e., given a member of the signal class $s(t) \in \mathcal{S}$ and

$$s_N(t) = \sum_{n=1}^N c_N \psi_N(t) \quad (3.5)$$

with the $\{c_N\}$ given by (1.19), then the error in the fit of $s_N(t)$ to $s(t)$ is

$$-T/2 \int_{-T/2}^{T/2} (s(t) - s_N(t))^2 dt = \sum_{N+1}^{\infty} c_N^2 \lambda_N. \quad (3.6)$$

The $\{\lambda_N\}$ sequence is seen to approach zero rapidly for sufficiently large N , thus (3.6) will be small for such N . Accuracy requirements and the behavior of the eigenvalues determine N . The behavior is discussed in [12], which makes the statement that, given a process time limited to a T second interval and band limited to $(-\alpha, \alpha)$, there are only $(\alpha T / \pi) + 1$ significant eigenvalues. Confirmation and more precise statements are in [8], [9], and [10]. The highest value of αT in this work is eight; therefore, the value $N = 4$ insures that all significant eigenvalues are included in the computations.

The robust filter performance shows a departure from trends established earlier by triangular and Gauss-Markov noise models for the parameter $(\sigma_0 / N_0 T)$. In Figure 3.1, the normalized distortion factor for a given fraction of signal distortion is seen to increase with increasing product (αT) . Note that, unlike the previous noise cases, the computations for σ_0 in the ideal bandlimited case use a pulsewidth which is constant for all values of αT . Thus, dissimilar performance measures are anticipated for this normalized parameter; yet Figure 3.1 is of interest because the equation (1.20) defining σ_0 represents the imposition of an energy restriction upon the signal of interest, resulting in a least favorable signal as defined in (1.21). It is seen that, for a given value of αT , the factor σ_0 is invariant to changes in E_0 ; i.e., changes in the energy

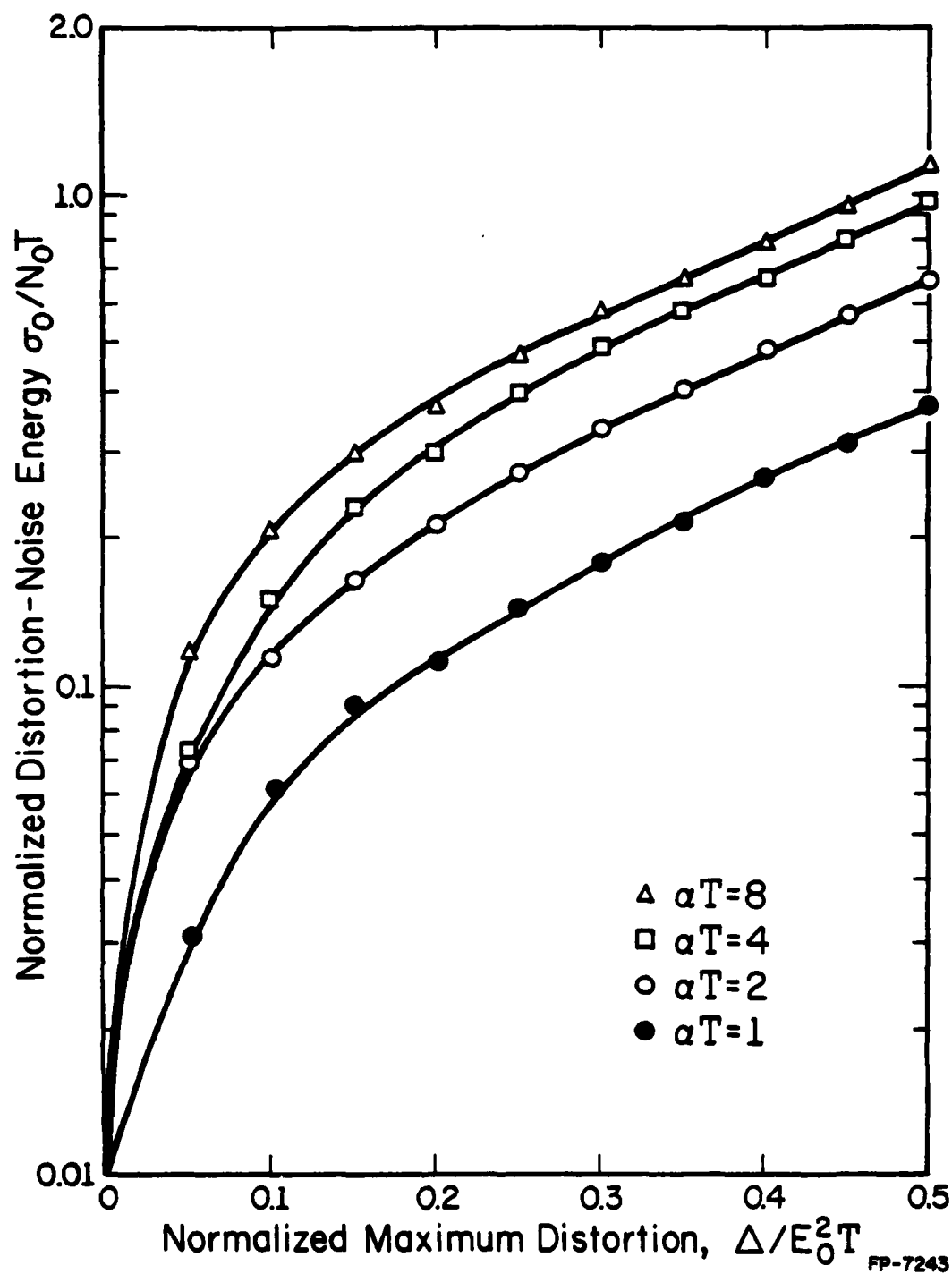


Figure 3.1. Distortion factor versus maximum signal distortion for ideally bandlimited noise.

$\|s^0\|^2$. It is apparent in Figure 3.1 that this energy restriction σ_0 varies with the product αT . More precisely, σ_0 is a function of the eigenvalues and of the terms $\{c_N^2\}$, which represent the distribution of signal energy in the eigenvectors.

The increase in a signal energy restriction with increase in αT should affect maximin error probabilities, and be apparent in a comparison of P_E attained by filters designed for noise models of different αT . The variation of P_E vs. αT shown in Figure 3.2, however, reveals that any effect on relative SNF by this trend in σ_0/N_0T vs. αT is masked by the singular characteristics of the zero-distortion filter. Figure 3.2 plots error probabilities for $SNR_0 = 10$ and $\alpha T = 1, 2, 4$, and 8. The calculations from $SNR = 20$ and 30 showed similar performance characteristics, namely that signal detection becomes increasingly sensitive to small amounts of channel distortion as αT decreases toward the narrowband case. For the bandlimited white noise process, a decrease in αT is a measure of the process' departure from a purely singular autocorrelation; therefore, severe performance degradation is predicted as the filter itself must become increasingly singular in nature.

The robust filter impulse response $h^R(t)$ is symmetric about the time interval midpoint, with evidence of suppressed singularities on interval endpoints. There is also evidence of local maxima and minima in the interval of the response, but the resolution of the tabulated data and graphs of the $\{\psi_N(t)\}$ functions did not allow plotting of $h^R(t)$ vs. time with sufficient accuracy for a detailed study of impulse response. Nor could data sources support closed form analytical expressions for the eigenfunctions

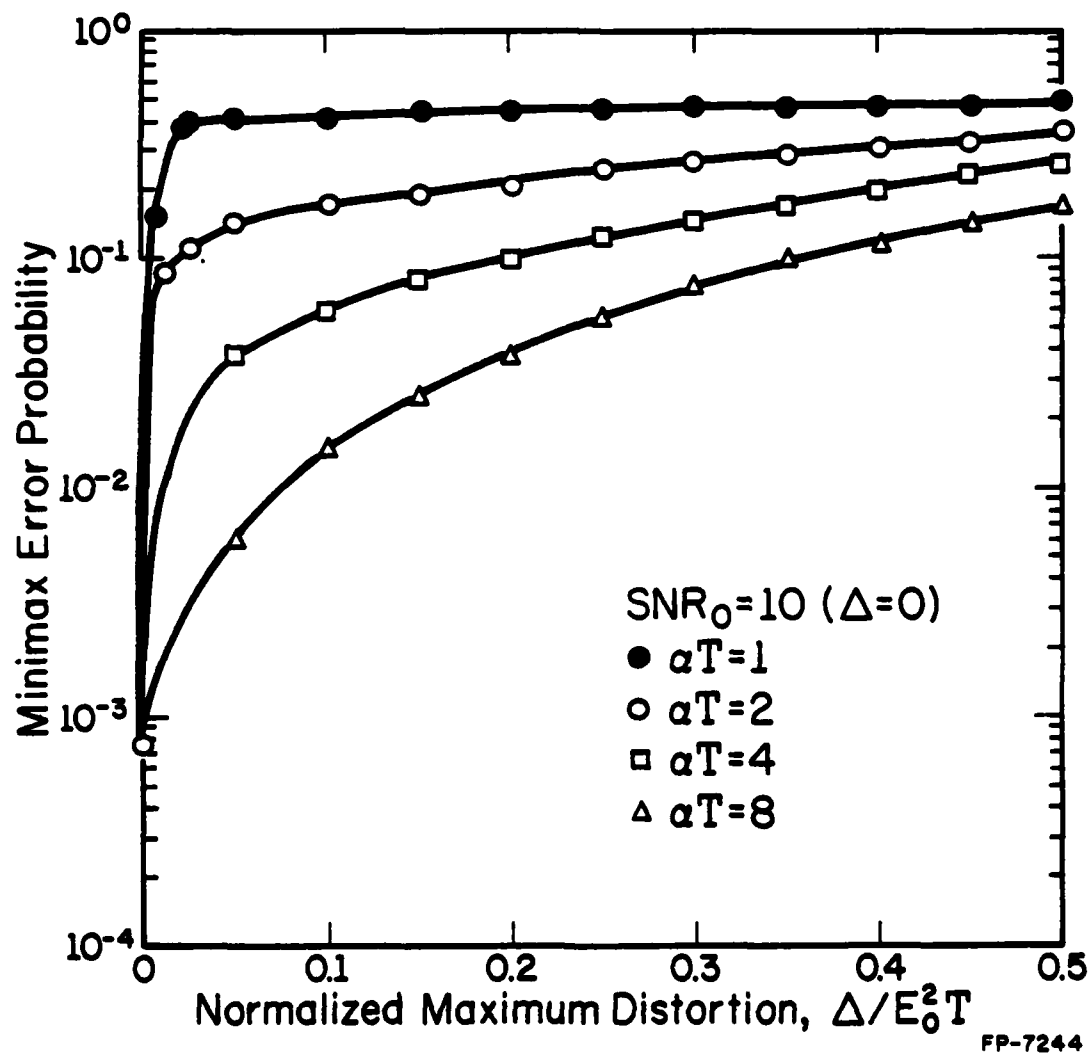


Figure 3.2. Maximin error probability versus maximum distortion for ideally bandlimited noise.

or the robust filter, prompting development of useful error bounds for calculations. Such bounds were supplied by: 1) assurance that four eigenvalues provide sufficient accuracy for the included values of αT ; 2) the proper truncation of the eigenvalue summations in the computations, providing reasonable lower and upper bounds for signal-to-noise ratio and maximum error probability, respectively. The truncation is as described in (2.21) and (2.22).

CHAPTER IV

THE WIENER NOISE PROCESS

The Wiener noise process was developed as a model for Brownian motion. It is of interest because a large class of processes are generated from the Wiener process: many others can be transformed into a Wiener process. As an example, a binary signal scheme is input to an integrator before transmission; it can be shown that if the signal is corrupted by a white noise process, e.g. thermal noise, the noise output of the integrator is

$$Y(t) = \int_0^t N(\tau) d\tau \quad (4.1)$$

where $N(t)$ is white noise with covariance

$$R_{NN}(\tau) = \sigma^2 \delta(\tau) . \quad (4.2)$$

$Y(t)$ is a sample function of a Wiener-Levy (or Wiener) process with the properties:

$$Y(0) = 0$$

$$E(Y(t)) = 0 \quad (4.3)$$

$$E[Y^2(t)] = \sigma^2 t \quad (4.4)$$

and the density function

$$p_Y[Y(t)] = (2\pi \sigma^2 t)^{-\frac{1}{2}} \exp[-Y^2(t)/(2\sigma^2 t)] . \quad (4.5)$$

The Wiener process, then, is a nonstationary random process. Using the above properties, the covariance kernel is found as

$$K_Y(t,u) = \sigma^2 \min(t,u) = \begin{cases} \sigma^2 t & , t \leq u \\ \sigma^2 u & , u \leq t \end{cases} \quad (4.6)$$

This function will be continuous on a given interval, thus the Karhunen-Loeve expansion exists for $0 \leq t, u \leq T$ in $[0, T]^2$. The equation of interest to generate the series expansion (1.15) becomes

$$\lambda \psi(t) = \sigma^2 \int_0^t u \psi(u) du + \sigma^2 t \int_t^T \psi(u) du \quad (4.7)$$

The solution sequences $\{\lambda_N\}$, $\{\psi_N(t)\}$, and $\{c_N\}$ needed to represent the robust filter for the Wiener noise process become

$$\lambda_N = 4N_0(T/\pi)^2 / (2N-1)^2 ; N = 1, 2, 3, \dots \quad (4.8)$$

$$\begin{aligned} \psi_N(t) &= \sqrt{2/T} \sin(\omega_N t) ; 0 \leq t \leq T \\ \omega_N^2 &= N_0 / \lambda_N \end{aligned} \quad (4.9)$$

The nominal signal is the integrator output

$$s_0(t) = E_0 t ; 0 \leq t \leq T \quad (4.10)$$

and the coefficients $\{c_N; N = 1, 2, 3, \dots\}$ become

$$c_N = \int_0^T E_0 t \psi_N(t) dt = \frac{-(-1)^N E_0 T^{3/2} 2^{5/2}}{\pi^2 (2N-1)^2} \quad (4.11)$$

We also have from (1.18) that

$$h^R(t) = \sigma_0^{-1} E_0 [(T-t) - 2T \sum_{n=1}^{\infty} k_N^{-1} \cos(\omega_N t)] \quad (4.12)$$

where

$$k_N = (1 + \sigma_0/\lambda_N)(N - \frac{1}{2})^2 \pi^2 . \quad (4.13)$$

The expressions for λ_N , $\psi_N(t)$, c_N , and $h^R(t)$ are uncomplicated, which will allow the robust filter performance to be derived analytically. An inspection of (4.12) and (4.13) reveals that the normalized impulse response $\sigma_0 E_0^{-1} h^R(t)$ will depend on the parameter $(\sigma_0/N_0 T)$ only. This distortion factor is specified in (1.20) and can be expressed analytically by defining a variable μ_Δ :

$$\mu_\Delta \in [0, 0.5]: \Delta = \mu_\Delta \int_0^T |s^0(t)|^2 dt ; \quad (4.14)$$

using the significant terms in the summation in (4.12), the following approximation determines σ_0 as a function of $(\Delta/\|s^0\|^2)$, where

$$\|s^0\|^2 = \int_0^T |s^0(t)|^2 dt = E_0^2 T^3/3 , \quad (4.15)$$

yielding

$$\sigma_0 \approx 4 N_0 T^2 / [\pi^2 (1 - \mu_\Delta)] . \quad (4.16)$$

The worst case filter performance is specified by the quantity $\langle h^R, s^L \rangle$.

This expression of SNR, derived from (1.21), becomes

$$\langle h^R, s^L \rangle \approx \sigma_0^{-1} \left[\frac{1}{3} E_0^2 T^3 (1 - \mu_\Delta) \right] . \quad (4.17)$$

Using (4.16), this performance measure becomes

$$\langle h^R, s^L \rangle \approx \pi^2 E_0^2 T (1 - \mu_\Delta)^2 / (12 N_0) ; \quad (4.18)$$

This result can be verified by considering the derivation for the zero distortion case; i.e., using equation (2.19), the limiting case as ($\Delta \rightarrow 0$)

$$\lim_{\Delta \rightarrow 0} \langle h^R, s^L \rangle \approx (c_1)^2 / \lambda_1 ; \quad (4.19)$$

only the first eigenvalue is considered significant. This approximation, which is derived independently, is practically identical to (4.17) for ($\mu_\Delta = 0$):

$$\begin{aligned} \langle h^R, s^L \rangle \Big|_{\Delta=0} &\approx (\pi^2/12) (E_0^2 T/N_0) \\ &\approx (c_1)^2 / \lambda_1 = (8/\pi^2) (E_0^2 T/N_0) . \end{aligned} \quad (4.20)$$

The derivation (4.17) has been proven accurate; however, the bounds for the approximation error for (4.18) are a function of the convergence of a p-series. The series' first term is proportional to the first eigenvalue and is seen to produce error bounds too large to allow confidence in computations using (4.17) or (4.18); consequently, the analytical derivation is useful only when general trends are to be investigated, or in the special case where the transmitted signal $s^0(t)$ is proportional to the N-th eigenfunctions of $\{R_N(t,u)\}$; e.g., $s^0(t) = a \psi_N(t)$. Poor [1] has shown that the solution to (1.14) becomes

$$h^R(t) = a \lambda_N^{-1} (1 + \sigma_0/\lambda_N)^{-1} \psi_N(T-t), \quad 0 \leq t \leq T ; \quad (4.21)$$

which is matched filter solution for coherent detection, with scale factor $(1 + \sigma_0/\lambda_N)^{-1}$. Such a solution would have no p-series involved in the calculations; thus, the results would possess both accuracy and tight error bounds based upon truncation of insignificant terms.

The robust filter performance in noise modelled as a Wiener process is demonstrated in Figures 4.1 and 4.2. These performance measures have error bounds obtained by truncation of the insignificant higher eigenvalues ($N > 20$). The parameter (σ_0/N_0T) is calculated as accurately as possible, and plotted in Figure 4.1 vs. $[\Delta/\|s^0\|^2]$. This term is then used in (1.22) to calculate a lower bound for worst performance $\langle h^R, s^L \rangle$. Finally, this lower bound for SNR is used in (1.13) to yield an upper bound for the maximin error probability, which is plotted in Figure 4.2 for values of $SNR_0 = 10, 20, \text{ and } 30$.

The performance of this filter continues the trends noted earlier for the robust filter. In particular, the detection method appears sensitive to the L_2 distortion, as SNR is greatly reduced for small fractions of signal structure distortion. The impulse response of $h^R(t)$ is essentially that of the matched filter for a signal defined as in (4.10); however, the increase in signal distortion produces a decrease in the peak amplitude of the response. This is similar to an estimation problem where the robust filter must operate on an increasingly distorted signal to obtain an estimate of the signal waveform with which to "match", for optimum SNR. The filter's confidence in the received waveform will diminish as distortion increases, and a signal estimate must be derived by the designed optimization scheme; e.g., minimum mean-square error point. The decrease in peak amplitude of a response $h^R(t)$ generated by such procedures thus yields evidence of both the desired signal structure and its increasing distortion by channel noise.

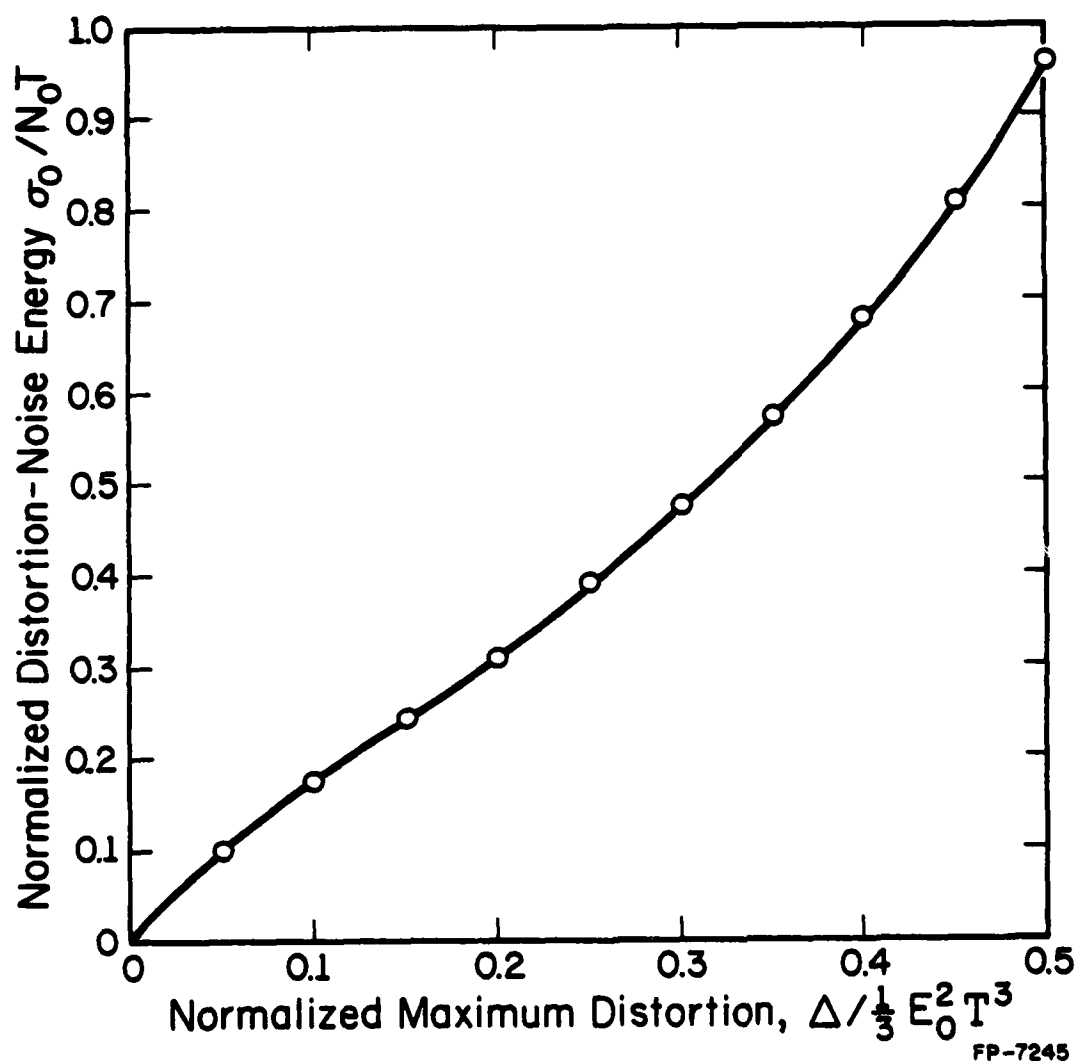


Figure 4.1. Distortion factor versus maximum signal distortion for the Wiener noise process.

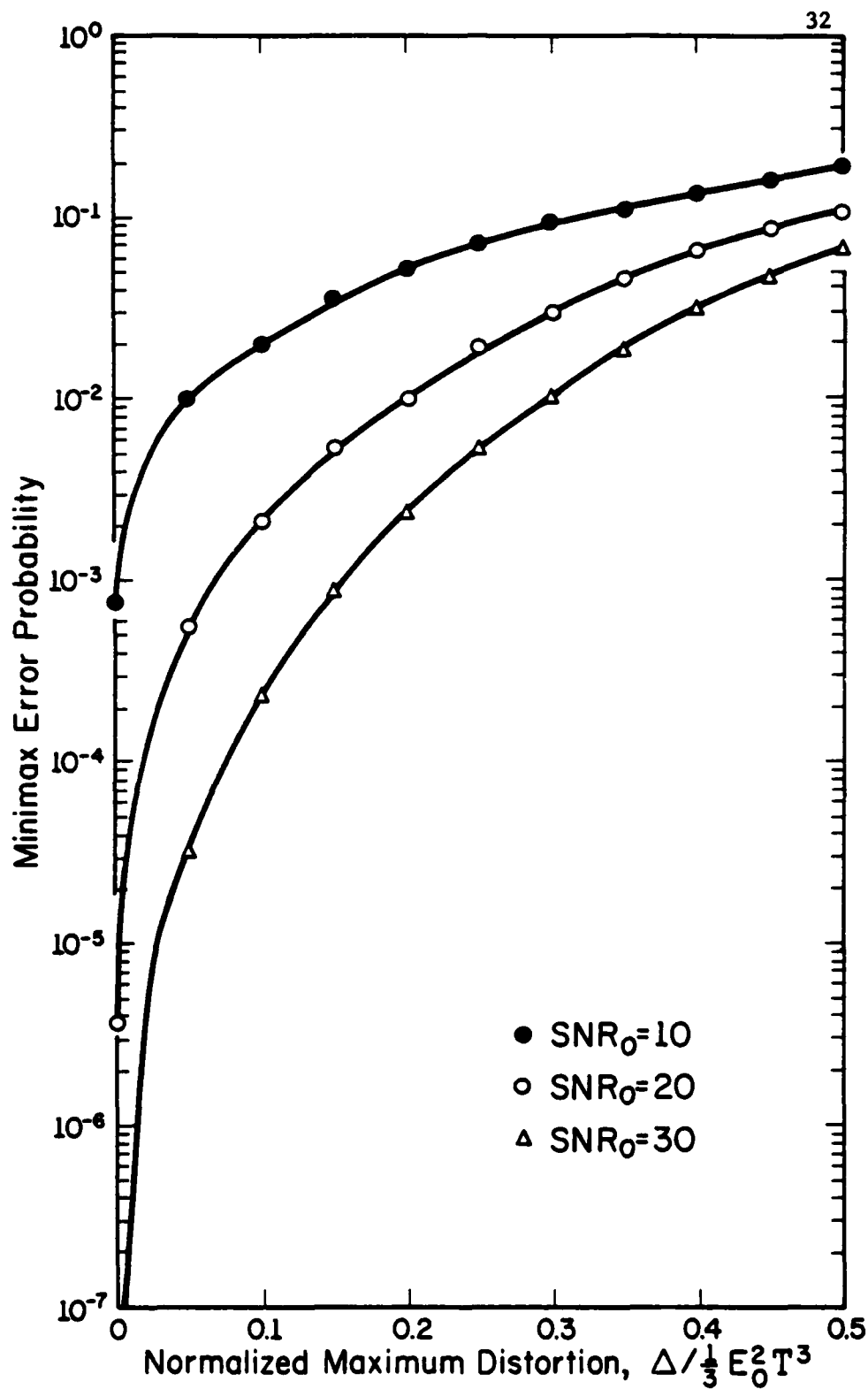


Figure 4.2. Maximin error probability versus maximum signal distortion for the Wiener noise process.

CHAPTER V

SUMMARY AND SUGGESTIONS

Matched filtering is seen to be sensitive to L_2 signal distortion when designed for three noise structures: the triangular-kernel noise, ideally bandlimited noise, and Wiener noise processes. It should be noted that pessimistic scenarios have been presented: 1) the quantity $\langle h^R, s^L \rangle$ specifying the worst performance was used to calculate signal-to-noise ratios; 2) in all cases, an upper bound for maximum error probability was plotted as the actual filter performance.

If the robust filter detection procedure is to be pursued, it would be advantageous to consider L_2 -norm distortion signal models such as (1.8) which limit the effective maximum Δ . Successful research in radar pulse design is an example of improved signal structure modelling.

In addition, future noise models to be considered might feature a singular (white noise) component in the noise autocorrelation function; e.g., if the noise structure is ideally bandlimited as in (3.1), define a new noise process

$$S_y(w) = S_x(w) + N_0/2 = \begin{cases} N_0/2 + S_0, & |w| < \alpha \\ N_0/2, & |w| > \alpha. \end{cases} \quad (5.1)$$

This technique may reduce the robust filter sensitivity to L_2 distortion, although the Mercer expansion of $R_Y(t,u)$ or the solution to the Karhunen-Loeve expansion might become difficult to determine.

REFERENCES

1. H. V. Poor, "On binary communication through a distorting channel," Proc. 12th Southeastern Symposium on System Theory, pp. 302-306, May 1980.
2. H. V. Poor, "A general approach to robust matched filtering," Proc. 22nd Midwest Symposium on Circuits and Systems, pp. 514-518, June 1979.
3. W. V. Lovitt, Linear Integral Equations. McGraw-Hill: New York, 1924.
4. A. Papoulis, Probability, Random Variables, and Stochastic Processes. McGraw-Hill: New York, 1965.
5. T. Kailath, "Some integral equations with 'nonrational' kernels," IEEE Transactions on Information Theory, vol. IT-12, no. 4, October 1966.
6. J. B. Thomas, An Introduction to Statistical Communication Theory. John Wiley and Sons: New York, 1969.
7. C. W. Helstrom, Statistical Theory of Signal Detection. Pergamon: Oxford, 1968.
8. D. Slepian and H. O. Pollak, "Prolate spheroidal wave functions, Fourier analysis and uncertainty - I," Bell System Tech. J., 40, pp. 43-64, 1961.
9. H. J. Landau and H. O. Pollak, "Prolate spheroidal wave functions, Fourier analysis and uncertainty - II," Bell System Tech. J., 40, pp. 65-84, 1961.
10. H. J. Landeau and H. O. Pollak, "Prolate spheroidal wave functions, Fourier analysis and uncertainty - III: The dimension of the space of essentially time- and band-limited signals," Bell System Techn. J., 41, p. 1295, 1962.
11. D. Slepian, "On bandwidth," Proc. IEEE, vol. 64, pp. 292-300, March 1976.
12. H. L. Van Trees, Detection, Estimation, and Modulation Theory - Part I. John Wiley and Sons: New York, 1968.
13. D. Slepian, "Estimation of signal parameters in the presence of noise," Trans. IRE, PGIT-3, p. 68, March 1954.

